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Gauge Gravitation Theory

Gennadi A. Sardanashvily[1]

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The particularity of the gauge gravitation theory is that Dirac fermion fields possess only Lorentz exact symmetries. It follows that different tetrad gravitational fields h define nonisomorphic representations γ_h of cotangent vectors to a space-time manifold X^4 by Dirac's γ-matrices on fermion fields. One needs these representations in order to construct the Dirac operator defined in terms of jet spaces. As a consequence, gravitational fields h fail to form an affine space modeled after any vector space of deviations $h' - h$ of some background field h. They therefore fail to be quantized in accordance with the familiar quantum field theory. At the same time, deformations of representation γ_h describe deviations σ of h such that $h + \sigma$ is not a gravitational field. These deviations form a vector space, i.e., satisfy the superposition principle. Their Lagrangian, however, differs from familiar Lagrangians of gravitation theory. For instance, it contains masslike terms.

1. INTRODUCTION

A gauge theory of space-time symmetries is reduced to the gauge theory of gravity by spontaneous symmetry breaking based on the fact that matter fields are Dirac fermion fields (Ivanenko and Sardanashvily, 1983; Sardanashvily and Zakharov, 1989). There are various spinor models of fermion matter. All observable fermion particles are Dirac fermions on which the Clifford algebra of Dirac's γ-matrices and the Dirac operator act. In fiber bundle terms, this means the following.

Let λ be a spinor bundle whose sections describe Dirac fermion fields ϕ. There is a vector fiber bundle λ_M with the structure Lorentz group $L = SO(3, 1)$ and the standard fiber which is the Minkowski space M so that the fiber-to-fiber morphism

$$\lambda_M \otimes \lambda \to \lambda$$

[1]Physics Faculty, Moscow University, 117234 Moscow, USSR.

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exists and defines the representation of elements of λ_M by Dirac's γ-matrices on ϕ. It is a key point that, to define the Dirac operator on ϕ, one must require λ_M to be isomorphic to the cotangent bundle T^*X over a space-time manifold X^4. Since the structure group of this bundle is $GL_4 = GL^+(4, \mathbb{R})$, such isomorphism takes place only in the following case. Given the principal linear frame bundle LX associated with T^*X, there is some reduced L-subbundle L^hX of LX associated with λ_M, that is, the equivalence principle holds. The corresponding tetrad gravitational field h turns T^*X into the fiber bundle in Minkowski space and yields the representation γ_h of cotangent vectors to X^4 (i.e., differential 1-forms on X^4) by Dirac's γ-matrices on sections of λ. These sections thereby describe Dirac fermion fields ϕ_h in the presence of the gravitational field h. The Higgs character of gravity issues from the fact that different gravitational fields h and h' define nonisomorphic representations γ_h and $\gamma_{h'}$. It follows that fermion fields must be considered only in a pair with a certain gravitational field. These pairs can be described by means of a spinor bundle over a generalized coordinate space $X^4 \times (GL_4/L)$. As a consequence, gravitational fields fail to form a vector space or an affine space modeled after any vector space of deviations of some background gravitational field, and so do not satisfy the superposition principle and therefore cannot be quantized in accordance with the conventional quantum field theory.

At the same time, one can consider deviations σ of h such that $h + \sigma$ is not a gravitational field (Sardanashvily and Zakharov, 1989). Such deviations are generated by non-Lorentz transformations of fibers of T^*X. The Dirac operator in the presence of deviations σ looks like that on a deformed manifold in the gauge theory of space-time translations (Sardanashvily and Gogberashvily, 1988; Sardanashvily, 1990). We use the Lagrangian of this theory in order to describe deviations σ. This Lagrangian differs from familiar Lagrangians of gravitation theory.

We assume that X^4 is an oriented paracompact connected smooth manifold and that it obeys the well-known topological conditions for the structure group of LX to be reducible to the Lorentz group L and can be extended to the structure group of a spinor fiber bundle.

2. GAUGE THEORY

In fiber bundle terms, matter fields ϕ are identified with global sections of a differentiable vector bundle

$$\lambda = (tl\lambda, \pi_\lambda, X, V, G)$$

with a total space $tl\lambda$, a base manifold X^4, canonical projection

$$\pi_\lambda: \quad tl\lambda \to X$$

a standard fiber V, and a structure Lie group G. We shall call λ a matter fiber bundle. Given the associated principal fiber bundle

$$\Lambda = (P_\Lambda, \pi, X, G)$$

with a total space P_Λ, the total space $tl\lambda$ of λ is defined to be the quotient $(P_\Lambda \times V)/G$ of $P_\Lambda \times V$ by identification of elements $(p \times v) \in P_\Lambda \times V$ with $(pg, g^{-1}v)$ for all $g \in G$. A section ϕ of λ is then determined by a V-valued equivariant function f_ϕ on P_Λ such that

$$\phi(\pi(p)) = [p]_V f_\phi(p), \qquad f_\phi(pg) = g^{-1} f_\phi(p), \qquad p \in P_\Lambda, \quad g \in G$$

where $[p]_V$ is the restriction of the canonical map $P_\Lambda \times V \to tl\lambda$ to the subspace $p \times V$.

Given a connection A in Λ with a connection 1-form A on P_Λ, the covariant differential $D\phi$ of sections ϕ of λ is defined as follows. For every vector field τ on X and its horizontal lift τ^H on P_Λ [i.e., $\pi_*(\tau^H) = \tau$, $A(\tau^H) = 0$ where π_* is the tangent map to π], the equivariant function $\tau^H f_\phi$ on P_Λ corresponds to the section

$$D_\tau \phi = (D\phi)(\tau)$$

of λ, which is the covariant derivative of ϕ along τ.

A Lagrangian L_ϕ of matter fields ϕ is defined to be a real function on the jet manifold $J_1\lambda$ of the fiber bundle λ (Mangiarotti and Modugno, 1990). Elements of $J_1\lambda$ are equivalent classes $j_x^1\phi$, $x \in X$, of sections ϕ such that ϕ and ϕ' belong to the same class $j_x^1\phi$ if and only if

$$\phi(x) = \phi'(x), \qquad \phi_*|_{T_xX} = \phi'_*|_{T_xX}$$

where T_xX is a tangent space to X at $x \in X$. Given a connection A in Λ, the covariant differential $D\phi$ of ϕ yields the mapping

$$J_1\lambda \to tl(T^*X \otimes \lambda) \tag{1}$$

where T^*X is a cotangent bundle. Coordinate atlases which define the structure of a differentiable manifold on $J_1\lambda$ are induced by atlases of λ and TX. The gauge principle requires a matter field Lagrangian to be invariant by transformations of these atlases.

We denote an atlas of λ by $\Psi^\lambda = \{U_i, \psi_i^\lambda\}$, where $\{U_i\}$ and ψ_i^λ are respectively an open covering of X and morphisms of trivialization of λ. An atlas Ψ^λ determines some reference frame in the sense that a section ϕ of λ can be expressed by a family of V-valued functions

$$\phi_i(x) = \psi_i^\lambda(x)\phi(x), \qquad x \in U_i$$

with respect to Ψ^λ. Given the associated principal fiber bundle Λ, we shall say that, for the same covering $\{U_i\}$, an atlas $\Psi^\Lambda = \{U_i, \psi_i^\Lambda\}$ of Λ and an atlas $\Psi^\lambda = \{U_i, \psi_i^\lambda\}$ of λ are associated if they are determined by the same family $\{z_i^\Lambda\}$ of local sections of Λ, that is,

$$z_i^\Lambda(\pi(p)) = p(\psi_i^\Lambda p)^{-1}, \qquad \psi_i^\lambda(x) = [z_i^\Lambda(x)]_V^{-1}, \qquad \pi(p) = x \in U_i$$

For instance, with respect to associated atlases Ψ^Λ and Ψ^λ,

$$\phi_i(x) = [z_i^\Lambda(x)]_V^{-1}\phi(x) = f_\phi(z_i^\Lambda(x)), \qquad x \in U_i$$

$$D\phi_i = (d - A_i)\phi_i$$

where $A_i = (z_i^\Lambda)^*A$ is a local connection 1-form whose coefficients are treated as gauge potentials.

An atlas $\Psi^T = \{U_i, \psi_i^T\}$ of the tangent bundle determines some space-time reference frame so that, given a fixed basis $\{t\}$ for the standard fiber $T = \mathbb{R}^4$ of TX, the vierbein

$$\{t_{ia}(x)\} = (\psi_i^T(x))^{-1}\{t_a\}, \qquad x \in U_i$$

associated with the atlas Ψ^T is erected at every point $x \in X$. The vierbein functions $t_i(x)$ play the role of local sections $z_i(x)$ of the principal fiber bundle LX. Atlases of the tangent bundle are equivalent to holonomic atlases

$$\Psi^T = \{U_i, \psi_i^T = (\chi_i)_*\}$$

correlating with coordinate atlases $\Psi_X = \{U_i, \chi_i\}$ of X. The associated vierbeins $t_\mu(x) = \partial_\mu$ are then oriented along coordinate curves.

Given a coordinate atlas Ψ_X, a holonomic atlas Ψ^T, and an atlas Ψ^λ, the jet manifold $J_1\lambda$ can be provided with local coordinates $(x_\mu, v_A, v_{A\mu})$ where v_A are coordinates in V with respect to some basis $\{v^A\}$ for V. For instance, a field $\phi(x)$ belongs to the classes $j_x^1\phi$, $x \in X$, with coordinates

$$(x^\mu, v_A = \phi_A(x), v_{A\mu} = \partial_\mu\phi_A(x))$$

and the mapping (1) reads

$$(x^\mu, v_A, v_{A\mu}) \to (x^\mu, (v_{A\nu} - A_\nu{}^B{}_A(x)v_B))$$

where $A_\nu{}^B{}_A$ are components of the local connection 1-form.

3. DIRAC FERMION FIELDS

We examine the case of Dirac fermion fields.

Let M be a Minkowski space with the Minkowski metric η. Consider the tensor algebra

$$A_M = \bigoplus_n M^n, \qquad M^0 = \mathbb{R}, \qquad M^{n>0} = \bigotimes^n M$$

of M. The complexified quotient of this algebra by the two-sided ideal generated by elements

$$e\otimes e'+e'\otimes e-\eta(e,e')\in A_M, \qquad e\in M$$

forms the complex Clifford algebra $C_{1,3}$. A spinor space V is defined to be a linear space of some minimal left ideal of $C_{1,3}$ on which this algebra acts on the left (Bugajska, 1986; Rodrigues and Figueiredo, 1990). We then have the representation

$$\gamma:\quad M\otimes V\to V \tag{2}$$

of elements of the Minkowski space $M\subset C_{1,3}$ by γ-matrices on V:

$$\hat{e}^a v^A=\gamma(e^a\otimes v^A)=\gamma^{aA}{}_B v^B$$

where $\{e^a\}$ is a fixed basis for M, $\{v^A\}$ is a fixed basis for V, and γ^a are Dirac's matrices of a fixed form.

Consider transformations which preserve representation (2). These are pairs (l, l_s) of Lorentz transformations l of the Minkowski space M and invertible elements l_S of $C_{1,3}$ such that

$$lM=l_sMl_s^{-1}, \qquad \gamma(lM\otimes l_sV)=l_s\gamma(M\otimes V) \tag{3}$$

These elements l_s form the Clifford group $G_{1,3}$. Action (3) of this group on M, however, is not effective. We restrict ourselves to the spinor subgroup $L_s\subset G_{1,3}$ keeping the standard Hermitian form on V, that is, $L=L_s/Z_2$.

Let λ_M be a fiber bundle with the structure group L, the standard fiber M, and the base X^4 so that the principal fiber bundle Λ_M associated with λ_M can be extended to a principal fiber bundle

$$\Lambda=(P_s, \pi, X^4, L_s)$$

with the structure group L_s:

$$tl\Lambda_M=P_s/Z_2, \qquad tl\lambda_M=(tl\Lambda_M\times M)/L=(P_s\times M)/L_s$$

Let $\lambda=(\pi_\lambda, X, V, L_s)$ be a fiber bundle associated with Λ. One can define the morphism

$$\gamma_\lambda:\quad tl(\lambda_M\otimes\lambda)=(P_s\times(M\otimes V))/L_s\to(P_s\times\gamma(M\otimes V))/L_s=tl\lambda$$

With respect to atlases $\Psi^{M\lambda}$ of λ_M and Ψ^λ of λ associated with atlases $\Psi^s=\{z_i^s\}$ of Λ and $\Psi=\{z_i^M=z_i^s/Z_2\}$ of Λ_M, this morphism reads

$$\hat{e}^a(x)v^A(x)=\gamma_\lambda(e^a(x)\otimes v^A(x))=\gamma^{aA}{}_B v^B(x), \qquad x\in U_i$$

Here

$$\{e^a(x)\}=\{(\psi_i^{M\lambda}(x))^{-1}e^a\}, \qquad \{v^A(x)=(\psi_i^\lambda(x))^{-1}v^A\}$$

are bases for fibers M_x and V_x of fiber bundles λ_M and λ, which are associated with atlases $\Psi^{M\lambda}$ and Ψ^{λ}, respectively.

Dirac fermion fields are described by global sections of the fiber bundle λ provided with the representation morphism γ_λ which yields the representation of sections of λ_M by γ-matrices on ϕ. To define the Dirac operator on these fields, one must require λ_M to be isomorphic to the cotangent bundle T^*X over X^4. This takes place only if the principal fiber bundle

$$LX = (P, \pi_{PX}, X, GL_4, \Psi)$$

contains a reduced subbundle L^hX with the structure group L.

4. GRAVITATIONAL FIELD

There is 1:1 correspondence between global sections h of the associated fiber bundle

$$\lambda_E = (\Sigma, \pi_{EX}, GL_4/L, GL_4)$$

with the standard fiber GL_4/L (we call h a tetrad field) and reduced L-subbundles L^hX of LX so that

$$\pi_{PE}P^h = h(\pi_{PX}p^h), \qquad P^h = tlL^hX$$

where π_{PE} is the canonical projection of P onto $E = tl\lambda_E = P/L$.

The fiber bundle λ_E is isomorphic to the fiber bundle of pseudo-Euclidean bilinear forms in cotangent spaces T_x^*X to X. A global section of this bundle is a pseudo-Riemannian metric g on X. A metric g can be represented as a nonvanishing global section of the tensor fiber bundle $TX \otimes TX$.

Given h and L^hX, let $\{z_i^h\}$ be a family of local sections of LX with values into P^h. These sections define an atlas Ψ^h of LX such that its transition functions are L-valued and, with respect to the associated atlas Ψ^{hT} of TX, metric functions of g come to the Minkowski metric:

$$g_i = \psi_i^{hT} g = \eta$$

If one provides the cotangent bundle T^*X with the fiber metric g and considers only atlases Ψ^h, this fiber bundle is endowed with the structure of an L fiber bundle M^hX in Minkowski spaces, that is,

$$tlT^*X = (P \times T^*)/GL_4 = (P^h \times M)/L$$

For different h and h', fiber bundles M^hX and $M^{h'}X$ are not isomorphic. Their fibers M_x and M'_x are cotangent spaces T_x^*X, but provided with different Minkowski space structures.

Since

$$h(x) = \pi_{PE}(z_i^h(x))$$

given the atlas Ψ^h and some holonomic atlas Ψ of LX, the tetrad field h can be uniquely represented by the family of tetrad functions

$$h_i(x) = \psi_i(x) z_i^h(x) = [z_i(x)]_T^{-1}[z_i^h(x)]_T, \qquad x \in U_i$$

Tetrad functions define gauge transformations of an atlas

$$\Psi^{hT} = \{U_i, \psi_i^{hT} = [z_i^h(x)]_T^{-1}\}$$

into the holonomic atlas

$$\Psi^T = \{U_i, \psi_i^T(x) = [z_i(x)]_T^{-1} = h_i(x)\psi_i^{hT}(x)\}$$

In the index form, tetrad functions (4) describe transformations

$$\begin{aligned} t_a^h(x) &= (\psi_i^{hT}(x))^{-1} t_a = (\psi_i^T(x))^{-1} h_a^\mu(x) t_\mu \\ &= h_a^\mu(x)(\psi_i^T(x))^{-1} t_\mu \\ &= h_a^\mu(x)\partial_\mu, \qquad x \in U_i \end{aligned}$$

between bases $\{\partial_\mu\}$ and $\{t_a^h(x)\}$ for tangent spaces and between the dual bases

$$\{dx^\mu\} = \{h_a^\mu(x) t^{ha}(x)\}$$

for cotangent spaces T_x^*X, which are associated with atlases Ψ and Ψ^h, respectively. For instance,

$$g_i = h_i\eta, \qquad g^{\mu\nu}(x) = h_a^\mu(x) h_b^\nu(x)\eta^{ab}$$

We say that a spinor fiber bundle λ_h describes Dirac fermion fields ϕ_h in the presence of a gravitational field h if the principal fiber bundle Λ associated with λ_h is the L_s-extension of the reduced L-subbundle $\Lambda_M = L^hX$. In this case, the fiber bundle λ_M is the cotangent bundle T^*X considered as the L fiber bundle M^hX. We then can define the representation

$$\gamma_h\colon \quad tl(T^*X \otimes \lambda_h) = (P^h \times (M \otimes V)/Z_2)/L \to (tl\Lambda \times \gamma(M \otimes V))/L_s = tl\lambda_h$$

of cotangent vectors to X by Dirac's matrices on elements of λ_h. With respect to an atlas $\Psi^s = \{U_i, z_i^s\}$ of Λ, associated atlases Ψ^λ of λ_h,

$$\Psi^h = \{U_i, z_i^h(x) = z_i^s(x)/Z_2\}$$

of LX and Ψ^{hT}, the morphism γ_h reads

$$\hat{t}^{ha}(x) v^A(x) = \gamma_h(t^{ha}(x) \otimes v^A(x)) = \gamma^{aA}{}_B v^B(x)$$

where $\{t^{ha}(x)\}$ and $\{v^A(x)\}$ are the corresponding bases for fibers T_x^*X and V_x.

In jet bundle terms, one can define the Dirac operator

$$L_D = \gamma_h D\colon \quad J_1\lambda_h \to tl(T^*X \otimes \lambda_h) \to tl\lambda_h$$

on fields ϕ_h. With respect to an atlas Ψ^h and some holonomic atlas, this operator reads

$$L_D\phi_h = \hat{d}x^\mu D_\mu \phi_h = h_a^\mu(x)\hat{t}^{ha}(x) D_\mu \phi_h = h_a^\mu(x)\gamma^a D_\mu \phi \tag{4}$$

where D_μ is the covariant derivative corresponding to some connection A in Λ.

Fermion fields ϕ_h and $\phi_{h'}$, are described by fiber bundles λ_h and $\lambda_{h'}$, so that the corresponding fiber bundles λ_M and λ'_M are associated with different L-subbundles L^hX and $L^{h'}X$ of LX. Fibers M_x and M'_x of λ_M and λ'_M are cotangent spaces T^*_xX, but provided with nonisomorphic structures of the Minkowski space. We shall compare representations γ_h and $\gamma_{h'}$.

For any two elements $p \in P^h_x$ and $p' \in P^{h'}_x$, there is an element $S \in GL_4$ such that

$$p' = pS, \qquad P^h_x = pL, \qquad P^{h'}_x = p'L = pSL$$

We then can write

$$\gamma_h: \quad t_x \otimes V_x = [p \times (t \otimes V/Z_2)]/L \to [p \times \gamma(t \otimes V)]/L_s = \gamma_h(t_x)V_x$$

$$\gamma_{h'}: \quad t_x \otimes V'_x = [p' \times (S^{-1}t \otimes V/Z_2)]/L \to [p' \times \gamma(S^{-1}t \otimes V)]/L_s = \gamma_{h'}(t_x)V'_x$$

If $h(x) \neq h'(x)$, we have $S \in GL_4 \backslash L$ and representations γ_h and $\gamma_{h'}$ fail to be isomorphic in the sense that there is no isomorphism ρ_V of the spinor space V such that

$$\gamma(S^{-1}M \otimes \rho_V V) = \rho_V \gamma(M \otimes V)$$

For instance, if $z^h_i(x) = p$ and $z^{h'}_i(x) = p'$, one can write

$$\begin{aligned} t(x) &= \tau_a(x)t^{ha}(x) = [z^h_i(x)L^{-1} \times L\tau_a(x)t^a]/L \\ &= [z^{h'}_i(x)L^{-1} \times LS^{-1a}_{\bar{b}}\tau_a(x)t^{\bar{b}}]/L = S^{-1a}_{\bar{b}}\tau_a(x)t^{h'\bar{b}}(x) \end{aligned}$$

where we span vierbeins $t^h(x)$ and $t^{h'}(x)$ by different indices a and $\bar{a}$, since these indices correspond to different metrics on X^4.

Since for different fields h and h', the representations γ_h and $\gamma_{h'}$ are nonisomorphic, Dirac fermion fields must be considered only in a pair with some gravitational field. A complex of such fermion-gravitation pairs can be described in the following way. For the sake of simplicity, we shall take the structure group of a spinor fiber bundle to be L.

The total space P of the principal fiber bundle LX is the total space of the principal fiber bundle Λ^L with the base $E = P/L$ and the structure Lorentz group L. Let

$$\lambda^L = (E, V, L)$$

be a spinor fiber bundle associated with Λ^L. The spinor fiber bundle λ_h then can be described as the one induced from λ^L by injection h of X onto $\pi_{PE}(P^h)$ in E. Therefore, each global section $\phi^L(\sigma)$ of λ^L, an atlas Ψ^L, and a connection A^L in Λ^L define, respectively, some global section

$$\phi_h(x) = \phi^L(h(x))$$

of λ_h, an atlas Ψ^λ, and a connection A^h in Λ. Conversely, for every h, there exists an open neighborhood U_E of the subset $h(X) \subset E$ so that the portion λ^L_U of λ^L over U_h is the pullback of λ_h with respect to projection π_{EX} of $U_h \subset E$ onto X. A global section ϕ_h of λ_h and a connection A^h in Λ then induce the pullback section

$$\phi^L(\sigma) = \phi_h(\pi_{EX}(\sigma)), \qquad \sigma \in U_h$$

of λ^L_U and the pullback connection A^L in λ^L_U. Remark that $U_h \neq E$, since the fiber bundle $GL_4 \to GL_4/L$ is not trivial.

Thus, pullback sections ϕ^L and pullback connections A^L can describe the above-mentioned fermion-gravitation complex. For instance, $A^L(t) = 0$ if t are tangent vectors to fibers $\pi^{-1}_{EX}(x) \subset E$, and there is an atlas Ψ^L_U of the fiber bundle λ^L_U so that field functions ϕ^L_i and A^L_i are constant on fibers $\pi^{-1}_{EX}(x) \subset E$. This means that, for any h' $[h'(X) \subset U_h]$, there exist atlases Ψ^h and $\Psi^{h'}$ of L^hX and $L^{h'}X$ so that

$$\phi_{hi}(x) = \phi^L_i(h(x)) = \phi^L_i(h'(x)) = \phi_{h'i}(x), \qquad A^h_i = A^{h'}_i$$

One therefore can vary independently gravitational potentials and fermion field functions in a matter field Lagrangian.

In terms of the fermion-gravitation complex, the Dirac operator can be described as follows. Let $J_1\lambda^L$ be the jet manifold of the fiber bundle λ^L. We define the operator

$$\tilde{L}_D\colon \quad J_1\lambda^L \to tl\lambda^L$$

such that, given the coordinate system on $J_1\lambda^L$ induced by an atlas Ψ^L of λ^L and a holonomic atlas Ψ^T, this operator takes the form

$$\tilde{L}_D\colon \quad \{x^\mu, h^\mu_a, \phi^L_A, \phi^L_{A\mu}\} \to h^\mu_a \gamma^{aB}_A \{\phi^L_{B\mu} - A^L_{\mu B}{}^C \phi^L_C\}$$

Here, $\{x^\mu, h^\mu_a\}$ are coordinates on the space E if an element $\sigma \in GL_4/L$ is replaced by its representer in GL_4. The Dirac operator (4) on fermion fields in the presence of a fixed gravitational field h is reproduced by restricting $\tilde{L}_D$ to the subspace $h(X) \subset E$.

5. DEVIATIONS OF A GRAVITATIONAL FIELD

In the conventional quantum field theory, to be quantized, fields must form a linear space, that is, they must satisfy the superposition principle.

By virtue of the specificity of a gravitational field h, tetrad functions $h^\mu_a(x)$ are written with respect to atlases Ψ^h defined by the field h itself. As a consequence, for different fields h and h', the indices a and $\bar{a}$ of tetrad functions h^μ_a and $h'^\mu_{\bar{a}}$ always correspond to different reference frames Ψ^h and $\Psi^{h'}$, and they are paired by different metrics on X^4. Tetrad functions thereby do not satisfy the superposition principle, and their deviations

$$h'^\mu_{\bar{a}} = (Sh)^\mu_{\bar{a}} = S^a_{\bar{a}} h^\mu_a \neq (\delta^a_{\bar{a}} + \varepsilon^a_{\bar{a}}) h^\mu_a, \qquad S \in GL_4 \backslash L \tag{5}$$

fail to be defined. Thus, in contrast with gauge potentials, tetrad gravitational fields h fail to form an affine space modeled after any vector space of deviations from some background field, and therefore they cannot be quantized.

If fermion fields are not considered, one usually chooses metric functions $g^{\mu\nu}$ as gravitational potential variables. Their small superposal deviations can be defined:

$$\begin{aligned} g'^{\mu\nu} &= (Sg)^{\mu\nu} = (e^\varepsilon g)^{\mu\nu} \approx g^{\mu\nu} + \varepsilon^{(\mu\nu)} \\ g'_{\mu\nu} &\approx g_{\mu\nu} - g_{\mu\alpha} g_{\nu\beta} \varepsilon^{(\alpha\beta)} = g_{\mu\nu} - \varepsilon_{(\mu\nu)} \end{aligned} \tag{6}$$

The metric function, however, fails to describe the space-time distributions which we need in quantum field theory. For different gravitational fields, there are space-time distributions which fail to be transformed into each other by Lorentz gauge transformations. Deviations of a gravitational field therefore cannot be neutralized by transformations of space-time distributions, and superposition of gravitational fields is accompanied by that of space-time distributions. We face such superposition in the case of gravitational singularities of the caustic type (Sardanashvily and Yanchevski, 1986).

We thus may conclude that superposal (quantum) deviations $\tilde{h} = h + \sigma$ of a geometrized gravitational field h (or g) cannot be geometrized fields which, for instance, would change reference frames Ψ^h defined by h. This is the characteristic feature of Higgs fields. In the axiomatic quantum field theory, different Higgs fields define nonequivalent representations of an algebra of matter fields. Quantum deviations of a Higgs field do not change a representation of this algebra, and so fail to result in some new Higgs field.

Tetrad functions h^μ_a in the Dirac operator (4) admit the following superposal deviations:

$$\tilde{h}^\mu_a = H_a{}^b h^\mu_b = (\delta^b_a + \sigma_a{}^b) h^\mu_b = H_\nu{}^\mu h^\nu_a = (\delta^\mu_\nu + \sigma_\nu{}^\mu) h^\nu_a = h^\mu_a + \sigma_a{}^\mu \tag{7}$$

$$L_D = \tilde{h}^\mu_a \gamma^a D_\mu \tag{8}$$

where $\tilde{h}^\mu_a$ is not a tetrad function because, in comparison with the expression (5), both indices a and b of $H_a{}^b$ correspond to the same reference frame associated with Ψ^h. In contrast with tetrad functions, we have

$$\tilde{h}^\mu_a \tilde{h}^a_\nu \neq \delta^\mu_\nu, \qquad \tilde{h}^\mu_a \tilde{h}^b_\mu \neq \delta^b_a, \qquad \tilde{h}^a_\mu = g_{\mu\nu}\eta^{ab}\tilde{h}^\nu_b$$

$$\tilde{g}^{\mu\nu} = \tilde{h}^\mu_a \tilde{h}^\nu_b \eta^{ab}, \qquad \tilde{g}_{\mu\nu} = \tilde{h}^a_\mu \tilde{h}^b_\nu \eta_{ab}, \qquad \tilde{g}^{\mu\nu}\tilde{g}_{\mu\alpha} \neq \delta^\nu_\alpha$$

The quantity $\tilde{g}$ is not a metric function. For instance, in comparison with relation (6), for small $\sigma^{\mu\nu} = \sigma_b{}^a h^\mu_a h^{b\nu}$, we have

$$\tilde{g}^{\mu\nu} \approx g^{\mu\nu} + \sigma^{\mu\nu}, \qquad \tilde{g}_{\mu\nu} \approx g_{\mu\nu} + g_{\mu\alpha}g_{\nu\beta}\sigma^{\alpha\beta}$$

We shall describe deviations (7) in the framework of the fiber bundle formalism. For every fiber P_x of LX, let us fix some element $p \in X$ and consider the following transformations of P_x:

$$P_x = pG^{-1} \to pH_x G^{-1}, \qquad H_x \in G = GL_4 \tag{9}$$

of P_x which is an isomorphism of the fiber bundle LX.

Transformation (9) yields the following mapping of the cotangent bundle $tlT^*X = (P \times T^*)/G$:

$$(pG^{-1} \times Gt)/G \to (pH_x G^{-1} \times GH_x^{-1}t)/G, \qquad t \in T^* \tag{10}$$

$$(pG^{-1} \times Gt)/G \to (pH_x G^{-1} \times Gt)/G = (pG^{-1} \times GH_x t)/G \tag{11}$$

The mapping (10) is the identity isomorphism of T^*X considered as the GL_4 fiber bundle. If $H_x \in GL_4\backslash L$, this mapping, however, is not a trivial mapping of the fiber bundle T^*X considered as different L fiber bundles λ_M. Let us assume that

$$p \in P^h_x, \qquad H_x \in G\backslash L, \qquad pH_x \in P^{h'}_x$$

for some h and h'. Morphism (10) then takes the form

$$\rho_1: \quad T^*_x X = (P^h_x \times M)/L = M_x \ni t(x) = (pL^{-1} \times Lt)/L$$

$$\to (pH_x L^{-1} \times LH_x^{-1}t)/L = t(x) \in M'_x = (P^{h'}_x \times M)/L = T^*_x X$$

and accompanies the transformation between gravitational fields h and h'.

The mapping (11) is an isomorphism of the cotangent bundle T^*X which, given the reference frame $z_i(x) = p$, is generated by transformations of cotangent spaces by means of operators H_x. If T^*X is considered as the L fiber bundle M^hX, this mapping can be written in two forms:

$$\rho_2: \quad M_x \ni (pL^{-1} \times Lt)/L \to (pL^{-1} \times LH_x t)/L$$

$$\rho_1\rho_2: \quad M_x \ni (pL^{-1} \times Lt)/L \to (pH_x L^{-1} \times Lt)/L$$

Given a gravitational field h and the corresponding representation γ_h, the mapping ρ_2 induces the representation mapping

$$\gamma_{h2}(t(x)) = \gamma_h(\rho_2 t(x))$$

In the index form, the mapping γ_{h2} reads

$$\begin{aligned}\gamma_{h2}:\quad & \tau_a t^{ha}(x)\otimes v(x) = [z_i^h(x)L^{-1}\times(LH_b{}^a(x)\tau_a\gamma^b\otimes(L_s v)/Z_2)]/L \\ & \to[z_i^h(x)L^{-1}\times L_s\gamma(H_b{}^a(x)\tau_a t^b\otimes v)/Z_2]/L = H_b{}^a(x)\tau_a\gamma^b v(x) \qquad (12)\end{aligned}$$

This mapping, like γ_h, defines the γ-matrix representation of cotangent vectors on spinor fields ϕ_h. Therefore, deviations

$$H_b{}^a = \delta_b^a + \sigma_b{}^a$$

and their superposition $\sigma + \sigma'$ can be defined. The Dirac operator corresponding to this representation γ_{h2} takes the form (8):

$$\begin{aligned} L_D &= \gamma_{h2}(dx^\mu)D_\mu\phi_h = h_a^\mu(x)\gamma_{h2}(t^{ha}(x))D_\mu\phi_h \\ &= h_a^\mu(x)H_b{}^a(x)\gamma^b D_\mu\phi_h = h_a^\mu(x)\gamma^a H_\mu{}^\nu(x)D_\nu\phi \qquad (13)\end{aligned}$$

Remark that, given a holonomic atlas, the functions $H_\mu{}^\nu(x)$ in expression (13) do not depend on a gravitational field, that is, gravitational potentials h_a^μ and deviations $\sigma_\nu{}^\mu$ are independent dynamic variables.

Deviations (7) and the Dirac operator (8) appear in the gauge theory of the translation group (Sardanashvily and Gogberashvily, 1988; Sardanashvily, 1990). We therefore may apply Lagrangians of this theory in order to describe fields σ. Note that, to construct a Lagrangian of deviations ε of a gravitational field g, one usually uses a familiar geometric Lagrangian of a field $g' = g - \varepsilon$ where g is treated as a background field. In the case of deviations (7), we can not follow this method because $\tilde{g}$ fails to be a true metric field.

Let AX be the fiber bundle of affine repers over a space-time manifold X^4. It is the principal fiber bundle with the affine structure group $A(4, \mathbb{R})$. For the sake of simplicity, AX is believed to be trivial. Provide the total space $Q = tlAX$ of AX with coordinates $\{x^\mu, u^a, S_a{}^\mu\}$. Here x^μ are coordinates in X^4, u^a are parameters of the translation subgroup T_4 of $A(4, \mathbb{R})$, and $S_a{}^\mu$ are coordinates of the reper $\{St_a\}$ with respect to the holonomic reper $\{\partial_\mu\}$, where $\{t_a\}$ is the fixed basis for T_4, and S is an element of the subgroup GL_4.

Remark that $\{x^\mu, u^\mu = S_a{}^\mu u^a\}$ are coordinates in the total space of the affine tangent bundle $A^T X$.

Let a general affine connection be in AX. Given the above-mentioned coordinates, its connection form A and the corresponding horizontal fields

τ^{HA} on Q read

$$A=(S^{-1})_\varepsilon{}^a(dS_b{}^\varepsilon+\Gamma_{\mu\alpha}{}^\varepsilon(x)S_b{}^\alpha\, dx^\mu)I_a{}^b+(du^a+B_\mu{}^a(x)\, dx^\mu)T_a$$
$$\tau^{HA}=\tau^\mu(x)\partial_\mu^{HA}=\tau^\mu(x)(\partial/\partial x^\mu-B_\mu{}^a(x)\partial/\partial u^a-\Gamma_{\mu\alpha}{}^\varepsilon(x)S_b{}^\alpha\partial/\partial S_b{}^\varepsilon) \tag{14}$$

where $I_a{}^b$ and T_a are generators of the group $A(4,\mathbb{R})$ and $\Gamma_{\alpha\mu}{}^\varepsilon$ are coefficients of a linear connection. In affine gauge theories, coefficients

$$B_\mu{}^\varepsilon=S_a{}^\varepsilon B_\mu{}^a$$

of the soldering form $B_\mu{}^\varepsilon\, dx^\mu$ on X^4 are treated as a gauge field of the translation group T_4. This tensor field defines the fiber-to-fiber morphism of tangent and cotangent bundles.

Let us consider the following mapping ρ of the space Q onto the total space P of the fiber bundle LX at points $S_a{}^\varepsilon u^a=u^\varepsilon(x)$:

$$\{x^\mu, u^a, S_a{}^\mu\}\to\{\xi^\mu(x^\varepsilon, u^\varepsilon(x)-S_a{}^\varepsilon u^a, 1), 0, S_a{}^\mu\}=\{x^\mu, 0, S_a{}^\mu\}$$

Here $\xi(x, u, s)$ is the geodesic defined by the linear connection Γ through the point x in the direction u, and $u(x)$ is some section of the fiber bundle A^TX. The tangent map ρ_* of the tangent bundle TQ over Q onto the tangent bundle TP over P transforms horizontal fields (14) on Q into fields

$$\begin{aligned}\tilde\tau^H&=\tau^\mu(x)(\delta_\mu^\nu+D_\mu^A u^\nu(x))[\partial/\partial x^\nu-\Gamma_{\nu\alpha}{}^\varepsilon(x)S_b{}^\alpha\partial/\partial S_b{}^\varepsilon]\\&=\tau^\mu(x)(\delta_\mu^\nu+D_\mu^A u^\nu(x))\partial_\nu^H=\tau^\mu(x)\tilde\partial_\mu^H\end{aligned} \tag{15}$$

on P. Here, by ∂_μ^H, we denote the horizontal lift of ∂_μ with respect to a linear connection, and

$$D_\mu^A u^\varepsilon(x)=\partial_\mu u^\varepsilon(x)+\Gamma_{\mu\alpha}{}^\varepsilon u^\alpha(x)+B_\mu{}^\varepsilon(x)=H_\mu{}^\varepsilon(x)-\delta_\mu^\varepsilon=\sigma_\mu{}^\varepsilon(x) \tag{16}$$

is the covariant derivative of fields $u(x)$. Remark that fields (15) are horizontal with respect to the linear connection Γ. A field $u(x)$, however, is always removed by gauge transformations. So only its covariant derivative (16) and fields σ can make physical sense.

Note that, in affine gauge theories, one usually considers the mapping

$$\beta:\quad \{x^\mu, u^a, S_a{}^\mu\}\to\{x^\mu, 0, S_a{}^\mu\}$$

of Q onto P. At points

$$S_a{}^\varepsilon u^a=u^\varepsilon(x)$$

we have $\beta=\rho$, but

$$\beta_*\tau^{HA}=\tau^\mu(x)[\partial/\partial x^\mu-\Gamma_{\mu\alpha}{}^\varepsilon S_b{}^\alpha\partial/\partial S_b{}^\varepsilon]\neq\rho_*\tau^{HA}$$

Let ϕ be some matter field on X^4 and f_ϕ be the corresponding equivariant function on $tl(LX\times\Lambda)$. We shall say that ϕ is defined on the deformed manifold X^4 if differentiation of ϕ is given by the expression

$$(\tilde D\phi)(\tau)=(df_\phi)(\tilde\tau^H)=(df_\phi)(\tau^\mu(x)H_\mu{}^\nu(x)\partial_\nu^H)$$

where ∂_ν^H is the horizontal lift of ∂_μ with respect to a connection in Λ. It follows that, in the field theory, deformation of a space-time manifold can be described by replacement of familiar covariant derivatives D_μ in the exterior differential $dx^\mu D_\mu$ by the quantities

$$\tilde{D}_\mu = (\delta_\mu^\alpha + \sigma_\mu{}^\alpha) D_\alpha = H_\mu{}^\alpha D_\alpha$$

For instance, the Dirac operator on the deformed manifold takes the form (13), and Lagrangians of gravitational and gauge fields are constructed by means of the modified curvature tensor

$$H_\mu{}^\varepsilon H_\nu{}^\beta R^{ab}_{\varepsilon\beta}$$

and the modified strength

$$H_\mu{}^\alpha H_\nu{}^\beta F^m_{\alpha\beta}$$

If one requires that the component $T^{00}_{(\sigma)}$ of a metric energy-momentum tensor of fields σ is positive, the Lagrangian of these fields can be chosen in the form

$$L_{(\infty)} = \tfrac{1}{2}[a_1 F_{\mu\nu}{}^\mu F^{\alpha\nu}{}_\alpha + a_2 F_{\mu\nu\sigma}(F^{\mu\nu\sigma} - 2F^{\nu\mu\sigma}) - \mu\sigma_\nu{}^\mu \sigma_\mu{}^\nu + \lambda \sigma_\mu{}^\mu \sigma_\nu{}^\nu]$$

$$F^\alpha{}_{\nu\mu} = D_{[\nu}\sigma_{\mu]}{}^\alpha, \qquad a_1 \geq 0, \quad a_2 \geq 0, \quad \mu > 0, \quad \lambda < \tfrac{1}{4}\mu$$

For instance, we obtain the following equation for a free weak field σ:

$$4a_2 \partial^\varepsilon (\omega_{\mu\varepsilon,\nu} + \omega_{\mu\nu,\varepsilon} - \omega_{\nu\varepsilon,\mu}) + 2a_1 \omega_{\alpha[\nu,\mu]}{}^\alpha - \mu\omega_{\mu\nu} = 0 \tag{17}$$

$$a_1 \mu^{-1}(\lambda - \mu)[\eta_{\mu\nu} \Box e - e_{,\mu\nu}] + 2a_1 \omega_{\alpha(\nu,\mu)}{}^\alpha - \mu e_{\mu\nu} + \lambda \eta_{\mu\nu} e = 0 \tag{18}$$

$$e_{\mu\nu} = \tfrac{1}{2}\sigma_{(\mu\nu)}, \qquad e = \sigma_\alpha{}^\alpha, \qquad \omega_{\mu\nu} = \tfrac{1}{2}\sigma_{[\mu\nu]}$$

If one takes the natural solution $\omega = 0$ of equation (17), equation (18) can be written in the form

$$e_{\mu\nu} = \frac{\mu - \lambda}{3\mu}(\eta_{\mu\nu} e - 3a_1 \mu^{-1} e_{,\mu\nu})$$
$$\Box e + m^2 e = 0, \qquad m^2 = \frac{\mu(\mu - 4\lambda)}{3a_1(\mu - \lambda)} \tag{19}$$

This equation admits plane wave solutions

$$e_{\mu\nu} = \frac{\mu - \lambda}{3\mu}\left(\eta_{\mu\nu} + \frac{\mu - 4\lambda}{\mu - \lambda}\frac{p_\mu p_\nu}{p^2}\right) a(p)\, e^{ipx}, \qquad p^2 = m^2 \tag{20}$$

Equation (19) and solutions (20) look promising in order to quantize the deviations (7) of a gravitational field.

Thus, we can say that transformation ρ_2 deforms the fibers of the cotangent bundle and thereby violates the identity of $T^*X = M^hX$ with the Minkowski space fiber bundle λ_M associated with the spinor bundle λ_h. In other words, the deviations (7) destroy the correlation of the Dirac fermion matter with the space-time geometric arena.

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